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Interaction of Intense Laser Beams with Electrons*

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The interaction of an intense coherent photon beam with free electrons is discussed. The photon beam is treated as a classical external electromagnetic field. The discussion is exact within the approximation of neglecting radiative corrections and the restriction to the case of a plane-wave field of arbitrary spectral composition and polarization properties. The scattering of a single photon out of a monochromatic beam by an isolated free electron is considered in detail. The cross sections corresponding to the scattering of the various harmonics of the incident beam are evaluated. These cross sections display a complicated dependence upon the intensity of the incident beam, at least for very intense beams. It is found that a mass change induced in the electron by the external field shifts the wavelength of the scattered photons by an amount depending on the intensity of the incident beam. Other processes involving free electrons in the final state are also considered briefly, and a discussion of the magnitude of the effects depending upon the intensity is given. Two Appendices are concerned with the electron Green's function and the vacuum-vacuum transformation function in the presence of a plane-wave field. In the course of the discussion of the latter, the problem of the correct definition of the vacuum current is encountered, and it is shown that a very careful procedure is necessary to obtain a covariant result.

1. INTRODUCTION

THE development of lasers has led to the availability, for the first time, of coherent photon beams of high intensity. Such beams give rise to a number of interesting and novel effects, depending nonlinearly on the beam intensity. The interaction of these beams with the elementary constituents of matter has not as yet been very fully studied.[1] In this paper, we shall concentrate on the problem of the interaction of a laser beam with a single free electron.[2] The methods used, can, however, be applied without difficulty to a considerable range of processes.

Since the number of photons in the laser beam is very large, it is a good approximation to use a semiclassical treatment in which the laser beam is treated as an external unquantized electromagnetic field. In this paper, we shall neglect radiative corrections; but if necessary they could easily be included in the usual way by making a perturbation expansion. We shall make the physically plausible idealization of representing the laser beam by a plane-wave field; that is, a field of arbitrary spectral composition and polarization properties, but characterized by a unique propagation direction. The calculations are then greatly simplified, for it has long been known that the Dirac equation in an external field of this form is exactly soluble.[3] We are therefore able to treat the external field exactly, without expanding in powers of the intensity.

As a preliminary to the main part of the discussion we give in Sec. 2 a straightforward derivation of the one-electron wave functions in the presence of a plane-wave field. These wave functions enter as essential constituents in a calculation of any process involving free electrons in the initial or final states.

As a specific example of the general technique, as well as for its intrinsic interest, we consider in Sec. 3 the problem of Compton scattering of a single photon out of the laser beam by an isolated free electron. We examine in detail the case of a monochromatic beam. In this case, as one would expect, the transition amplitude decomposes into a sum of incoherent amplitudes corresponding to the various harmonics of the incident beam. The corresponding cross sections are evaluated, and for a beam of very high intensity (photon densities of the order of 10^{27} cm^{-3}) are found to exhibit a complicated dependence on the intensity and polarization of the beam. Such intensities are probably well beyond the reach of forseeable experimental techniques, and the primary experimental interest would be in a nonrelativistic situation. We give, therefore, explicit expressions only for the nonrelativistic limit of the cross sections including terms of zeroth and first order in the intensity. Thus we obtain the first correction to the Thompson scattering cross section, and the leading term of the cross section for first harmonic photon production. There is, however, another effect, an intensity-dependent wavelength shift of the scattered

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[1] Those phenomena which have been studied include the generation of harmonics and of beat frequencies. A treatment of harmonic generation using the usual methods of perturbation theory has been given by Z. Fried, Nuovo Cimento **22**, 1303 (1961). The generation of beats has been discussed from a microscopic viewpoint by Z. Fried and W. M. Franck, Nuovo Cimento **27**, 218 (1963). See also Z. Fried and H. M. Cha, Phys. Letters **1**, 220 (1962). Another nonlinear effect of great theoretical interest, but much harder to observe experimentally, is the scattering of light by light, which has been discussed in this context by J. McKenna and P. M. Platzman, Phys. Rev. **129**, 2354 (1963).

[2] This problem has already been studied using quite different methods by Z. Fried, Phys. Letters **3**, 349 (1963). It was this paper which supplied the principal motivation for the present work.

[3] D. M. Volkov, Z. Physik **94**, 250 (1935).

photon, which may well be observable with a more moderate increase in currently available laser-beam intensities. This remarkable effect results from a change in the mass of an electron when it propagates in a plane-wave field.

The major results of Sec. 3 are of a purely classical nature. It has recently come to our attention that a discussion of the scattering processes considered in that section has been given by Vachaspati,[3a] using classical electrodynamics. His results differ somewhat from ours, and we discuss the relationship between the two in Appendix C in the course of a general examination of the classical aspect of these processes. We also outline in this Appendix an alternative classical treatment which corresponds closely to the quantum-mechanical derivation of Sec. 3 and confirms the results given there.

The intensity-dependent parameters in the various Compton scattering cross sections would be large if these processes involved large changes in the electron velocity. We therefore consider, in Sec. 4, the possibility that the presence of a laser beam of moderate intensity might lead to large effects in processes, such as β decay, which involve high-velocity electrons.[4] We find, however, that there is a closure relation which implies that the effects are still of the same order of magnitude as in the case of Compton scattering.

In Appendix A we give a derivation of the electron Green's function in a plane-wave field.[5] This Green's function is used in Appendix B to show that in such a field the vacuum-to-vacuum transformation function is precisely unity. This result, which is used in the text, would be expected on physical grounds, and has been established previously for the linearly polarized case by Schwinger,[6] using rather more indirect arguments. In the course of the discussion, however, we encounter a problem of more general interest, namely that of the correct definition of the vacuum expectation value of the current operator. We show that a covariant, and in fact vanishing, vacuum current can only be obtained by a proper definition of this quantity as the response to a change in the external field, and by a very careful consideration of the limiting process involved in the definition of bilinear combinations of the field operators. We hope, therefore, that this discussion may throw some light on this general problem.

2. THE WAVE FUNCTIONS

We are interested in laser beams of very high intensity. It is therefore a good approximation to treat the beam as a classical external field[7] $A_\mu(x)$. We shall consider the idealization of representing this field as a plane-wave field; that is, a field of arbitrary spectral composition and polarization properties, but which is distinguished by a unique propagation direction. This direction may be covariantly characterized by a null vector n_μ lying on the forward light cone,[8]

$$n^2=0\,,\quad n_0>0\,.$$

We shall use the notation

$$y=n\cdot x=n_\mu x^\mu\,, \tag{2.1}$$

so that our plane-wave vector potential is an arbitrary function of y, $A_\mu(y)$. In virtue of the null property of n_μ, any such function satisfies the wave equation

$$\partial^2 A(y)=n^2(d/dy)^2 A(y)=0\,.$$

We shall choose a class of gauges such that the vector potential satisfies the condition

$$n\cdot A=0\,. \tag{2.2}$$

There still exists the freedom of making gauge transformations of the type

$$A_\mu(y)\rightarrow A_\mu(y)+\partial_\mu\lambda(y)=A_\mu(y)+n_\mu(d/dy)\lambda(y)\,.$$

The field strength tensor is given by

$$F_{\mu\nu}=\partial_\nu A_\mu-\partial_\mu A_\nu=n_\nu(d/dy)A_\mu-n_\mu(d/dy)A_\nu\,. \tag{2.3}$$

Now, if we calculate the transition probability for any process involving electrons, or other charged particles, we shall need the appropriate wave functions for these particles in the presence of the external field. It is therefore convenient to compute these wave functions here, as a preliminary to the calculations given later.

Since the Dirac equation in second-order form differs only by a spin term from the Klein-Gordon equation, it is useful to consider first the wave function for an incoming scalar particle. This function is defined by[9]

$$\Phi_p^{\text{in}}(x)=\langle 0|\phi(x)|\mathbf{p},\text{in}\rangle_A\,, \tag{2.4}$$

[3a] Vachaspati, Phys. Rev. **128**, 664 (1962) and Errata **130**, 2598 (1963).

[4] The possibility that beta decay rates might be appreciably altered by the presence of a laser beam was suggested to us by G. Feldman and P. T. Matthews.

[5] This Green's function has previously been calculated, in a somewhat implicit form, by J. Schwinger, Phys. Rev. **82**, 664 (1951) for the particular case of a linearly polarized field.

[6] J. Schwinger, Ref. 5.

[7] Alternatively, one may represent the initial (final) state of the laser beam as an eigenstate of the positive frequency part of the incoming (outgoing) photon field, $|A_\mu^{(+)\prime}\rangle$. Such classical limit states have been used extensively by J. Schwinger, Phys. Rev. **91**, 728 (1953) and **92**, 1283 (1953), and have recently been applied by R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963) to problems of coherence involving laser beams. [See also E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963) and R. J. Glauber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963).] It is not hard to see that, with the neglect of radiative corrections and of the depletion of the initial state when a small number of photons are scattered out of it, this formalism is equivalent to the one used in the present paper.

[8] We employ natural units ($c=\hbar=1$), and a metric such that

$$n\cdot x=n_\mu x^\mu=n_0x_0-\mathbf{n}\cdot\mathbf{x}.$$

[9] In the case of a general external field, the quantity of physical interest is $\langle 0,\text{out}|\phi(x)|\mathbf{p},\text{in}\rangle_A$. For a plane-wave field, however, we need not distinguish between the incoming and outgoing vacuum states, since we show in Appendix B that in this case the vacuum-to-vacuum transformation function $\langle 0,\text{out}|0,\text{in}\rangle_A$ is

where the subscript A denotes that this quantity is evaluated in the presence of the external field $A_\mu(x)$. It satisfies the Klein-Gordon equation

$$(\Pi^2-m^2)\Phi_p{}^{\text{in}}(x)=0,$$

and the initial condition

$$\Phi_p{}^{\text{in}}(x)\sim e^{-ip\cdot x} \quad \text{as} \quad x_0\to-\infty.$$

Here Π_μ is the gauge-covariant derivative

$$\Pi_\mu=i\partial_\mu-eA_\mu.$$

Recalling that A_μ is a function of y only, we are led to look for a solution of the form

$$\Phi_p{}^{\text{in}}(x)=e^{-ip\cdot x}f(y),$$

with the initial condition

$$f(y)\to 1 \quad \text{as} \quad y\to-\infty.$$

Because of the null character of n, and the relation $p^2=m^2$, the equation for $f(y)$ is

$$[i(d/dy)-I_p(y)]f(y)=0,$$

where

$$I_p(y)=(1/2n\cdot p)[2ep\cdot A(y)-e^2A^2(y)]. \tag{2.5}$$

This equation has an obvious solution, which gives

$$\Phi_p{}^{\text{in}}(x)=e^{-ip\cdot x}\exp\left[-i\int_{-\infty}^{y}dy'I_p(y')\right]. \tag{2.6}$$

By an entirely similar argument, or by using time-reversal invariance, we find that the wave function for an outgoing particle,

$$\Phi_p{}^{\text{out}}(x)^*=\langle\mathbf{p},\text{out}|\phi^\dagger(x)|0\rangle_A, \tag{2.7}$$

is given by

$$\Phi_p{}^{\text{out}}(x)^*=e^{ip\cdot x}\exp\left[-i\int_{y}^{\infty}dy'I_p(y')\right]. \tag{2.8}$$

We now return to the case of the Dirac equation. The wave function of an incoming electron of momentum p and spin λ,

$$\Psi_{p\lambda}{}^{\text{in}}(x)=\langle 0|\psi(x)|\mathbf{p}\lambda,\text{in}\rangle_A, \tag{2.9}$$

satisfies the second-order Dirac equation[10]

$$(\gamma\cdot\Pi+m)(\gamma\cdot\Pi-m)\Psi_{p\lambda}{}^{\text{in}}(x) =[\Pi^2-m^2+\tfrac{1}{2}e\sigma F(x)]\Psi_{p\lambda}{}^{\text{in}}(x)=0,$$

and the initial condition

$$\Psi_{p\lambda}{}^{\text{in}}(x)\sim e^{-ip\cdot x}u_{p\lambda} \quad \text{as} \quad x_0\to-\infty.$$

Here

$$\tfrac{1}{2}\sigma F=\tfrac{1}{2}\sigma_{\mu\nu}F^{\mu\nu}=-i(\gamma\cdot n)\gamma\cdot(d/dy)A(y).$$

Note that since $n^2=n\cdot A=0$, this matrix satisfies

$$\sigma F(y)\sigma F(y')=0.$$

In view of the expression obtained above for the Klein-Gordon wave function, we look for a solution of the form

$$\Psi_{p\lambda}{}^{\text{in}}(x)=g(y)e^{-ip\cdot x}\exp\left[-i\int_{-\infty}^{y}dy'I_p(y')\right]u_{p\lambda},$$

where $g(y)$ is a matrix function of y satisfying the initial condition

$$g(y)\to 1 \quad \text{as} \quad y\to-\infty.$$

Substituting this expression in the second-order Dirac equation, we obtain for $g(y)$ the equation

$$[2\,in\cdot p(d/dy)+\tfrac{1}{2}e\sigma F(y)]g(y)=0,$$

which has the solution, satisfying the prescribed initial condition,

$$g(y)=\exp\left[\frac{ie}{4n\cdot p}\int_{-\infty}^{y}dy'\sigma F(y')\right] =1+\frac{e}{2n\cdot p}\gamma\cdot n\gamma\cdot A(y).$$

Thus, finally,[11]

$$\Psi_{p\lambda}{}^{\text{in}}(x)=[1+(e/2n\cdot p)\gamma\cdot n\gamma\cdot A(y)] \times e^{-ip\cdot x}\exp\left[-i\int_{-\infty}^{y}dy'I_p(y')\right]u_{p\lambda}. \tag{2.10}$$

Similarly, the wave-function for an outgoing electron of momentum p and spin λ,

$$\bar\Psi_{p\lambda}{}^{\text{out}}(x)=\langle\mathbf{p}\lambda,\text{out}|\bar\psi(x)|0\rangle_A, \tag{2.11}$$

is given by

$$\bar\Psi_{p\lambda}{}^{\text{out}}(x)=\bar u_{p\lambda}[1+(e/2n\cdot p)\gamma\cdot A(y)\gamma\cdot n] \times e^{ip\cdot x}\exp\left[-i\int_{y}^{\infty}dy'I_p(y')\right]. \tag{2.12}$$

3. HIGH-INTENSITY COMPTON SCATTERING

In this section, we consider the scattering of a single photon out of an intense plane-wave beam incident on an isolated free electron. There is no difficulty in going through the usual limiting arguments with wave pockets, provided that we initially take the incident beam to be a wave train of finite duration and only later go to the limit of a monochromatic beam. We therefore take the initial and final electron states to be states of definite momentum and spin, $\mathbf{p}$, λ and $\mathbf{p}'$,

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precisely unity. Our method of calculation would be inapplicable if this were not so, since we should not be able to specify the initial condition at $x_0\to-\infty$.

[10] We use Dirac matrices defined by $\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}$ and $\sigma_{\mu\nu}=\frac{1}{2}i[\gamma_\mu,\gamma_\nu]$.

[11] Since $\Psi_{p\lambda}{}^{\text{in}}$ satisfies the second-order Dirac equation, and initial conditions which themselves satisfy the first-order Dirac equation, the expression $\chi=(\gamma\cdot\Pi-m)\Psi_{p\lambda}{}^{\text{in}}$ satisfies the equation $(\gamma\cdot\Pi+m)\chi=0$ and vanishes in the limit $x_0\to-\infty$. Thus χ is identically zero, and $\Psi_{p\lambda}{}^{\text{in}}$ automatically satisfies the first-order Dirac equation.

λ', and consider the scattered photon to be in a state of definite momentum and polarization, $\mathbf{k}'$, ϵ'. The application of the usual reduction procedure gives

$$\langle \mathbf{p}'\lambda',\mathbf{k}'\epsilon', \text{out}|\mathbf{p}\lambda, \text{in}\rangle_A$$
$$= -i\int (dx)e^{ik'\cdot x}\langle \mathbf{p}'\lambda', \text{out}|\epsilon'\cdot j(x)|\mathbf{p}\lambda, \text{in}\rangle_A ,$$

where

$$j_\mu(x) = e\bar{\psi}(x)\gamma_\mu\psi(x) .$$

With the neglect of radiative corrections, the two-electron Green's function factorizes into a product of one-electron Green's functions. Accordingly,[12]

$$\langle \mathbf{p}'\lambda',\mathbf{k}'\epsilon', \text{out}|\mathbf{p}\lambda, \text{in}\rangle_A$$
$$= -ie\int (dx)e^{ik'\cdot x}\bar{\Psi}_{p'\lambda'}{}^{\text{out}}(x)\epsilon'\cdot\gamma\Psi_{p\lambda}{}^{\text{in}}(x) , \quad (3.1)$$

where the wave functions are those defined in the preceding section. We thus obtain, within the framework of quantum field theory, the familiar result of the first-quantized theory which expresses the transition amplitude as a wave function matrix element of the interaction term $e^{ik'\cdot x}\epsilon'\cdot\gamma$ representing the scattered photon.

On inserting the expressions (2.10) and (2.12) for these wave functions, we obtain

$$\langle \mathbf{p}'\lambda',\mathbf{k}'\epsilon', \text{out}|\mathbf{p}\lambda, \text{in}\rangle_A$$
$$= -ie\int (dx)\, \exp[i(p'+k'-p)\cdot x]$$
$$\times \exp\left[-i\int_y^\infty dy' I_{p'}(y') - i\int_{-\infty}^y dy' I_p(y')\right]$$
$$\times \bar{u}_{p'\lambda'}[1+(e/2n\cdot p')\gamma\cdot A(y)\gamma\cdot n]\gamma\cdot\epsilon'$$
$$\times[1+(e/2n\cdot p)\gamma\cdot n\gamma\cdot A(y)]u_{p\lambda} . \quad (3.2)$$

We may now go to the limit of a monochromatic plane wave. For the general case of arbitrary polarization, we may write this in the form

$$A_\mu(y) = \text{Re}(\mathfrak{a}_\mu e^{-i\omega y}) = \text{Re}(\mathfrak{a}_\mu e^{-ik\cdot x}) , \quad (3.3)$$

where $\mathfrak{a}_\mu$ is a complex vector specifying the amplitude and polarization of this field, ω is its real angular frequency (in the frame in which $n_0=1$), and

$$k_\mu = \omega n_\mu \quad (3.4)$$

is the corresponding energy-momentum vector. In this limit, there is a divergence in the amplitude. However, the divergent part is merely a constant phase factor which has no physical effect and may therefore be omitted. It arises of course because we have chosen an infinitely long wave train. With the omission of this phase factor, we then find the expression

$$\langle \mathbf{p}'\lambda',\mathbf{k}'\epsilon', \text{out}|\mathbf{p}\lambda, \text{in}\rangle_A$$
$$= -ie\int (dx)\, \exp[i(p'+k'-p)\cdot x]$$
$$\times \exp[i\xi\sin(k\cdot x-\alpha)$$
$$+\tfrac{1}{2}i\eta\sin(2k\cdot x-2\alpha-\beta)+i\zeta k\cdot x]$$
$$\times \bar{u}_{p'\lambda'}[1+(e/2k\cdot p')\gamma\cdot\text{Re}(\mathfrak{a}e^{-ik\cdot x})\gamma\cdot k]\gamma\cdot\epsilon'$$
$$\times[1+(e/2k\cdot p)\gamma\cdot k\gamma\cdot\text{Re}(\mathfrak{a}e^{-ik\cdot x})]u_{p\lambda} , \quad (3.5)$$

where the gauge-invariant, dimensionless parameters ξ, η, ζ, α, β are given by

$$\xi e^{i\alpha} = e\left(\frac{p'\cdot\mathfrak{a}}{p'\cdot k} - \frac{p\cdot\mathfrak{a}}{p\cdot k}\right) , \quad (3.6)$$

$$\eta e^{i(2\alpha+\beta)} = \frac{e^2}{4}(-\mathfrak{a}^2)\left(\frac{1}{p'\cdot k} - \frac{1}{p\cdot k}\right) , \quad (3.7)$$

$$\zeta = \frac{e^2}{4}(-\mathfrak{a}\cdot\mathfrak{a}^*)\left(\frac{1}{p'\cdot k} - \frac{1}{p\cdot k}\right) . \quad (3.8)$$

We note that the phase α cannot be physically significant, since it may be absorbed by a phase change of $\mathfrak{a}$. It is convenient to note here that the kinematical relations obtained below show that

$$[(1/p'\cdot k)-(1/p\cdot k)] \geqslant 0 ,$$

whence for all polarization states

$$\zeta \geqslant \eta \geqslant 0 .$$

The limiting cases are $\eta=\zeta$ for linear polarization, and $\eta=0$ for circular polarization.

The exponential factor involving the sine functions is periodic in $k\cdot x$, and in order to perform the integration over x it is useful to express it as a Fourier series. To this end, we use the generating function of Bessel functions

$$e^{iz\sin\theta} = \sum_{r=-\infty}^{\infty} e^{ir\theta}J_r(z) .$$

Then we obtain

$$\exp\{i\xi\sin(\theta-\alpha)+\tfrac{1}{2}i\eta\sin[2(\theta-\alpha)-\beta]\}$$
$$= \textstyle\sum_r e^{-ir(\theta-\alpha)}C_r(\xi,\eta,\beta) , \quad (3.9)$$

where

$$C_r(\xi,\eta,\beta) = \textstyle\sum_s e^{is\beta}J_{r-2s}(-\xi)J_s(-\tfrac{1}{2}\eta) . \quad (3.10)$$

These functions C_r share many of the properties of the Bessel functions of which they are composed. In the

[12] In the general case, there would be an additional term involving the vacuum current and an additional factor of the vacuum-vacuum transformation function in this equation. We may omit these for the case of a plane-wave field in view of the results obtained in Appendix B, compare footnote 9.

particular case of a circularly polarized beam, they reduce to Bessel functions,

$$C_r(\xi,0,\beta)=J_r(-\xi).$$

The application of the Fourier inversion formula to the defining relation (3.9) yields

$$C_r(\xi,\eta,\beta)=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\exp[ir\theta+i\xi\sin\theta+\tfrac{1}{2}i\eta\sin(2\theta-\beta)]. \quad (3.11)$$

Upon integrating this relation by parts, we obtain a relation among contiguous functions,

$$2rC_r+\xi(C_{r-1}+C_{r+1})+\eta(e^{i\beta}C_{r-2}+e^{-i\beta}C_{r+2})=0. \quad (3.12)$$

When we substitute the series (3.9) in the matrix element, and perform the integration over x, we obtain a sum of terms involving energy-momentum δ functions, of the form

$$\langle \mathbf{p}'\lambda',\mathbf{k}'\epsilon',\text{out}|\mathbf{p}\lambda,\text{in}\rangle_A = -i\sum_r(2\pi)^4\delta(p'+k'-p-[r-\zeta]k)T_r. \quad (3.13)$$

This structure displays the decomposition of the complete scattering amplitude into incoherent amplitudes which describe the scattering of the various harmonics of the incident beam. The energy-momentum conservation equation

$$p'+k'=p+rk-\zeta k \quad (3.14)$$

contains an unfamiliar term, ζk. This term arises from the altered propagation character of the electron in the laser beam. The Green's function obtained in Appendix A shows that for large times the electron propagates in the beam with a mass $m^2+\Delta m^2$, where Δm^2 is positive, and for a monochromatic beam is given by

$$\Delta m^2=\tfrac{1}{2}e^2(-\mathcal{A}\cdot\mathcal{A}^*). \quad (3.15)$$

As the electron propagates into the beam, its effective mass changes from m^2 to $m^2+\Delta m^2$, and it is plausible that the only component of its momentum which can change during this process is that along the direction of k. Thus the effective momentum of the electron inside the beam is

$$\bar{p}=p+(\Delta m^2/2k\cdot p)k,\quad \bar{p}^2=m^2+\Delta m^2. \quad (3.16)$$

Since ζ may be written as

$$\zeta=(\Delta m^2/2k\cdot p')-(\Delta m^2/2k\cdot p),$$

the energy-momentum conservation equation may be written in the form

$$\bar{p}'+k'=\bar{p}+rk, \quad (3.17)$$

which expresses the conservation of momenta inside the beam. We note that this equation can only be satisfied if r is a positive integer.

The amplitudes T_r are given by

$$T_r=e^{ir\alpha}\sum_{s=-2}^{2}C_{r-s}(\xi,\eta,\beta)M_s, \quad (3.18)$$

where the M_s are spinor matrix elements which can be written, after some elementary spinor algebra, in the form

$$M_0=e\bar{u}_{p'\lambda'}[\gamma\cdot\epsilon'+\zeta(k\cdot\epsilon'/k\cdot k')\gamma\cdot k]u_{p\lambda}, \quad (3.19)$$

$$M_1=\tfrac{1}{4}e^2e^{-i\alpha}\bar{u}_{p'\lambda'}\left[\frac{\gamma\cdot\mathcal{A}\gamma\cdot k\gamma\cdot\epsilon'}{k\cdot p'}+\frac{\gamma\cdot\epsilon'\gamma\cdot k\gamma\cdot\mathcal{A}}{k\cdot p}\right]u_{p\lambda}, \quad (3.20)$$

$$M_{\pm2}=\tfrac{1}{2}e\eta e^{\pm i\beta}(k\cdot\epsilon'/k\cdot k')\bar{u}_{p'\lambda'}\gamma\cdot k u_{p\lambda}, \quad (3.21)$$

with M_{-1} obtained from M_1 by the replacement $e^{-i\alpha}\mathcal{A}\rightarrow e^{i\alpha}\mathcal{A}^*$. The amplitude T_r is clearly invariant under a gauge transformation of the external field,

$$\mathcal{A}\rightarrow\mathcal{A}+\lambda k.$$

Its invariance under a transformation

$$\epsilon'\rightarrow\epsilon'+\lambda'k'$$

of the polarization vector of the scattered photon can be easily verified with the help of the relation (3.12) among contiguous C_r functions.

It is clear from an inspection of the matrix elements M_s that it is advantageous to use a gauge such that $k\cdot\epsilon'=0$, for then $M_{\pm2}$ vanish and M_0 takes on a particularly simple form. However, we shall need the results of our computation in other gauges. Hence we shall not fix the gauge at this stage, but exploit the gauge invariance of T_r by replacing ϵ' by the gauge invariant quantity

$$\bar{\epsilon}_\mu'=\epsilon_\mu'-\left(\frac{k\cdot\epsilon'}{k\cdot k'}\right)k_\mu'. \quad (3.22)$$

It is then a straightforward matter to sum and average over the electron spins. We obtain

$$\langle|T_r|^2\rangle_{\text{av}}=\tfrac{1}{2}\sum_\lambda\sum_{\lambda'}|T_r|^2 =\tfrac{1}{4}e^2[|A_r|^2+(k\cdot k'/m^2)|B_r|^2], \quad (3.23)$$

where

$$A_r=(1/m)[2p\cdot\bar{\epsilon}'C_r-e(e^{-i\alpha}\bar{\epsilon}'\cdot\mathcal{A}C_{r-1}+e^{i\alpha}\bar{\epsilon}'\cdot\mathcal{A}^*C_{r+1})], \quad (3.24)$$

and

$$|B_r|^2=2(r-\zeta)|C_r|^2+\zeta(|C_{r-1}|^2+|C_{r+1}|^2)+\xi\,\text{Re}[(C_{r-1}+C_{r+1})C_r^*]+2\eta\,\text{Re}[e^{i\beta}C_{r-1}C_{r+1}^*]. \quad (3.25)$$

Note that the entire dependence on the polarization vector ϵ' appears in the expression A_r. Since $\bar{\epsilon}'$ is gauge invariant, we may easily obtain polarization sums by allowing ϵ' to range over the four coordinate directions. Thus

$$\sum_{\epsilon'} a\cdot\bar{\epsilon}'\bar{\epsilon}'\cdot b=-a_{\perp}\cdot b_{\perp}=-a\cdot b+\frac{a\cdot kk'\cdot b+a\cdot k'k\cdot b}{k\cdot k'},$$

where $a_{\perp}$ denotes the projection of a in the plane orthogonal to k and k'. Using the identity

$$p\cdot k'=(r-\zeta)p'\cdot k,$$

which follows at once from the momentum conservation equation, we then find[13]

$$\sum_{\epsilon'} |A_r|^2=4\left[\frac{p'\cdot kp\cdot k}{m^2k'\cdot k}|B_r|^2-|C_r|^2\right],$$

and hence

$$\sum_{\epsilon'} \langle |T_r|^2\rangle_{\rm av}=e^2\left[\frac{(p'\cdot k)^2+(p\cdot k)^2}{2m^2k'\cdot k}|B_r|^2-|C_r|^2\right]. \quad (3.26)$$

It is convenient to introduce at this point a dimensionless parameter ν^2, independent of the momenta, which characterizes the intensity of the incident beam. We define

$$\nu^2=\frac{\Delta m^2}{m^2}=\frac{e^2}{m^2}(-\tfrac{1}{2}\mathfrak{a}\cdot\mathfrak{a}^*)>0. \quad (3.27)$$

It is easily seen that the squared matrix element $\langle|T_r|^2\rangle_{\rm av}$ contains only ν^{2r} and higher powers. It is convenient to extract this factor, and define Q_r by

$$\langle|T_r|^2\rangle_{\rm av}=\tfrac{1}{4}e^2\nu^{2r}Q_r, \quad (3.28)$$

$$Q_r=\left|\frac{A_r}{\nu^r}\right|^2+\frac{k\cdot k'}{m^2}\left|\frac{B_r}{\nu^r}\right|^2. \quad (3.29)$$

Before proceeding with an explicit calculation of the various cross sections, it is useful to examine the magnitude of the effects we are considering. If we revert temporarily to conventional units, we may write the parameter ν^2 in the form

$$\nu^2=(2\pi^2)^{-1}\alpha\rho\lambda\lambda_C{}^2, \quad (3.30)$$

where $\alpha=e^2/4\pi\hbar c=1/137$, ρ is the photon number density of the laser beam, λ is the wavelength of the beam, and $\lambda_C=h/mc$ is the electron Compton wavelength. This shows that the relevant quantity is the number of photons in a cylinder of radius λ_C, and length λ. Inserting the value of λ_C, we obtain

$$\nu^2\approx2\times10^{-23}\rho\lambda,$$

if ρ is in cm^{-3} and λ in cm. Thus for a wavelength of the order of 5000 Å, we would require a photon density of the order of 10^{27} cm^{-3} to obtain a parameter $\nu^2\approx1$. Unfortunately, the photon densities of conventional laser beams are several orders of magnitude less than this, and it seems rather unlikely that densities of this order will be attained in the near future. However, one might hope to attain a photon density of, say, 10^{23} cm^{-3}, so that $\nu^2\approx10^{-4}$. This may still give rise to some observable effects, which will be discussed below. An alternative form for ν^2 is

$$\nu^2=\mathcal{E}\lambda^2r_0/\pi mc^2, \quad (3.31)$$

where $r_0=e^2/4\pi mc^2$ is the classical electron radius, and $\mathcal{E}$ is the electromagnetic energy density in the laser beam. This form exhibits ν^2 as the ratio of the electromagnetic energy contained in a volume λ^2r_0 to the rest energy of the electron. It thus demonstrates that ν^2 is a purely classical quantity, independent of Planck's constant $\hbar$. It also shows that to attain $\nu^2\approx1$ we would require an energy flux of the order of 3×10^{10} W per square wavelength.[14]

We now return to the calculation of the various cross sections. We shall now restrict our discussion to the laboratory frame in which the electron is initially at rest. The differential cross section for the scattering of the $(r-1)$th harmonic is

$$\frac{d\sigma_r}{d\Omega}=\frac{1}{\rho}\frac{d\Phi_r}{d\Omega}\langle|T_r|^2\rangle_{\rm av}, \quad (3.32)$$

where $\rho=(e^2/m^2\omega\nu^2)^{-1}$ is the flux of the incident photon beam, and Φ_r is the phase space of the final state, which we now proceed to calculate. We may write this quantity in the form

$$\Phi_r=\frac{2m}{(2\pi)^2}\int(d_4k')\delta(k'^2)\theta(k_0')\int(d_4p')\delta(p'^2-m^2)\theta(p_0')\times\delta_4(p'+k'-p-[r-\zeta]k). \quad (3.33)$$

The evaluation of this integral requires some care, because ζ is itself a function of p' through the relation

$$\zeta=\frac{m^2\nu^2}{2}[(1/p'\cdot k)-(1/p\cdot k)].$$

The simplest way to proceed is to separate the integrations over the components of p' along k and k' from the remainder. Using the notation $p_{\perp}'$, introduced above, for the projection of p' in the plane perpendicular

[13] It is clear from this expression that $|B_r|^2$ is indeed a positive quantity.

[14] It is well to note here that although ν^2 apparently diverges with increasing wavelength, this divergence cannot be obtained in a physically realizable situation. For, in order that the field be a good approximation to a monochromatic plane wave, it must extend at least over a volume of the order of λ^3, and the energy within this volume, $\mathcal{E}\lambda^3$, must be finite. Thus, as the wavelength increases without bound, ν^2 must in fact vanish as $1/\lambda$.

to k and k', we have

$$\int (d_4p')\delta(p'^2-m^2)\theta(p_0')\delta_4(p'+k'-p-[r-\zeta]k)$$
$$=\int (d_2p_\perp')d(p'\cdot k)d(p'\cdot k')$$
$$\times\delta[p_\perp'^2+2(p'\cdot kp'\cdot k'/k\cdot k')-m^2]\theta(p'\cdot k)$$
$$\times\delta_2(p_\perp'-p_\perp)\delta(p'\cdot k+k'\cdot k-p\cdot k)$$
$$\times\delta(p'\cdot k'-p\cdot k'-[r-\zeta]k\cdot k').$$

The Jacobian of the transformation is easily seen to be cancelled by the factor arising from transforming the δ functions. The θ function $\theta(p_0')$ has been replaced by $\theta(p'\cdot k)$ since for a vector p' lying on the hyperboloid $p'^2=m^2$ the conditions $p_0'>0$ and $p'\cdot k>0$ are equivalent. It is then a straightforward matter to perform the p' integrations successively. We obtain the expression

$$\Phi_r=\frac{2m}{(2\pi)^2}\int (d_4k')\delta(k'^2)\theta(k_0')\theta(k\cdot p-k\cdot k')$$
$$\times\delta\{2k'\cdot p+(m^2\nu^2k\cdot k'/k\cdot p)-2r[k\cdot p-k\cdot k']\},$$

from which it is obvious that the argument of the θ function is positive if r is positive. We may now evaluate this integral in the laboratory frame in which

$$p\cdot k=m\omega,\quad p\cdot k'=m\omega',\quad k\cdot k'=2\omega\omega'\sin^2\tfrac{1}{2}\theta,$$

where ω and ω' are the energies corresponding to k and k' and θ is the scattering angle of the photon. The vanishing of the argument of the δ function gives the relation between the incident and scattered frequencies,

$$\omega'=\omega_r'=\frac{mr\omega}{m+(2r\omega+m\nu^2)\sin^2\frac{1}{2}\theta}. \tag{3.34}$$

On performing the integration over k' we obtain

$$\Phi_r=\frac{\omega_r'^2}{8\pi^2r\omega}\int d\Omega, \tag{3.35}$$

and hence

$$d\sigma_r/d\Omega=(\nu^{r-1}r_0)^2(\omega_r'^2/2r\omega^2)Q_r, \tag{3.36}$$

where, as before, r_0 is the classical electron radius.

The expression we have obtained exhibits, for extremely high intensities ($\nu^2\approx1$), a very complex dependence on the intensity, and on the scattering angle and polarization vectors. Before examining the simpler approximate forms which suffice for the description of possible experiments, we wish to discuss the frequency shift implied by the general relation (3.34) between the incident and scattered frequencies. This relation is simpler if re-expressed as a relation between the wavelengths,

$$\lambda_r'=(1/r)\lambda+[2\lambda_C+(\nu^2/r)\lambda]\sin^2\tfrac{1}{2}\theta. \tag{3.37}$$

For $r=1$ and $\nu^2=0$, this reduces of course to the usual formula for the Compton effect. The additional term proportional to ν^2 arises from the altered propagation character of the electron in a plane-wave field which was discussed above. Note that, despite the fact that $\Delta m^2>0$, the effect is to make the electron appear lighter, for λ_C is replaced by the larger quantity $\lambda_C+\nu^2\lambda/2r$. This apparent paradox arises from the directional nature of the effect of the altered mass. In terms of the momenta $\bar{p}$ and $\bar{p}'$ inside the beam the momentum conservation equation

$$\bar{p}'+k'=\bar{p}+rk$$

is precisely that for ordinary Compton scattering with an incident photon of momentum rk. Hence the frequencies in the rest frame of $\bar{p}$ are related by the usual Compton formula with a Compton wavelength corresponding to the mass $(m^2+\Delta m^2)^{1/2}$. However the effect of a Lorentz transformation from the rest-frame of $\bar{p}$ to that of p is to change this relation into the one quoted above. As we discussed above, it is probably not experimentally feasible to attain a value of ν^2 much in excess of say 10^{-4}. Although such a small value of this parameter does not lead to very significant effects elsewhere, it might well give an observable wavelength shift, since such shifts can be measured with great accuracy.

We now return to the evaluation of the cross sections, restricting our attention to the experimentally important case of linear polarization, and to the radiation gauge. Then $\mathcal{A}_\mu$ is of the form

$$\mathcal{A}=(0,a\boldsymbol{\varepsilon}), \tag{3.38}$$

where

$$\boldsymbol{\varepsilon}^2=1,$$

and the phase β is zero. The remaining parameters occurring in the rth amplitude are

$$\xi_r=\frac{r\nu\sqrt{2}}{1+\nu^2\sin^2\frac{1}{2}\theta}\hat{k}'\cdot\boldsymbol{\varepsilon}, \tag{3.39}$$

$$\eta_r=\zeta_r=\frac{r\nu^2\sin^2\frac{1}{2}\theta}{1+\nu^2\sin^2\frac{1}{2}\theta}, \tag{3.40}$$

where $\hat{k}'$ is a unit vector in the direction of $\mathbf{k}'$. We note that these parameters depend only on ν^2 and the angles involved in the process, and not explicitly upon the frequency of the incident beam, ω. The amplitude A_r can be expressed in the form

$$A_r=2^{1/2}\nu\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'(C_{r-1}+C_{r+1})+(\hat{k}\cdot\boldsymbol{\varepsilon}'/\sin^2\tfrac{1}{2}\theta)$$
$$\times[C_r+(\nu/\sqrt{2})\hat{k}'\cdot\boldsymbol{\varepsilon}(C_{r-1}+C_{r+1})], \tag{3.41}$$

and hence, recalling the definition (3.25) of B_r, it is clear that both these amplitudes are independent of ω. Thus it follows from the expression (3.29) for Q_r that the nonrelativistic limit[15] is obtained simply by omitting the contribution of B_r, and we secure[16]

$$\left(\frac{d\sigma_r}{d\Omega}\right)_{\rm NR}=(\nu^{r-1}r_0)^2\frac{r}{2(1+\nu^2\sin^2\frac{1}{2}\theta)^2}\left|\frac{A_r}{\nu^r}\right|^2. \quad (3.42)$$

As we have discussed above, the parameter ν^2 is quite small for any foreseeable experimental situation. We shall therefore expand our results and retain only terms up to order ν^2. This procedure yields a ν^2 correction to the familiar Thompson scattering cross section and the leading term of the cross section for the first harmonic production. With the help of the expansions

$$C_0=1-\tfrac{1}{4}\xi^2+O(\nu^4),$$
$$C_1=-\tfrac{1}{2}\xi(1+\tfrac{1}{4}\eta-\tfrac{1}{8}\xi^2)+O(\nu^5),$$
$$C_2=-\tfrac{1}{4}(\eta-\tfrac{1}{2}\xi^2)+O(\nu^4),$$

we obtain

$$\left(\frac{d\sigma_1}{d\Omega}\right)_{\rm NR}\approx r_0^2[(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}')^2-\tfrac{1}{2}\nu^2(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}')\{(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}')$$
$$\times[5\sin^2\tfrac{1}{2}\theta+(\hat{k}'\cdot\boldsymbol{\varepsilon})^2]-(\hat{k}'\cdot\boldsymbol{\varepsilon})(\hat{k}\cdot\boldsymbol{\varepsilon}')\}], \quad (3.43)$$

and

$$\left(\frac{d\sigma_2}{d\Omega}\right)_{\rm NR}\approx(\nu r_0)^2[2(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}')(\hat{k}'\cdot\boldsymbol{\varepsilon})+\tfrac{1}{2}(\hat{k}\cdot\boldsymbol{\varepsilon}')]^2. \quad (3.44)$$

4. OTHER PROCESSES

We have found in the previous section that the nonlinear intensity effects in Compton scattering are very small unless laser beams of extreme intensity are employed. It is natural to ask whether it is not possible to find some alternative process for which these effects would be appreciable even for beams of moderate intensity. The parameter in Compton scattering,

$$\xi=e|(p'\cdot\mathfrak{a}/p'\cdot k)-(p\cdot\mathfrak{a}/p\cdot k)|,$$

is small because the change in the electron's velocity is small. It is of the order[17] $e\mathfrak{a}/\omega$ multiplied by the change of the electron's velocity, whose order of magnitude is the small quantity ω/m. Thus we are led to consider whether a process which involves high-velocity electrons within a monochromatic plane-wave field would exhibit a large effect. We shall consider a general class of processes of this kind, but it may be helpful to keep a specific example in mind. Such an example is provided by beta decay[4] in which a fast electron is emitted and the only other charged particle is slowly moving.

The presence of spin terms is an inessential complication, and for laser beams of moderate intensity these terms are in any case unimportant; they are smaller by a factor of order $e\mathfrak{a}/m$ than those which are independent of spinor matrices. Thus we shall consider a process in which a charged scalar particle is emitted, and which therefore involves the wave function $\Phi_{p'}{}^{\rm out*}(x)$ calculated in Sec. 2. The complete matrix element for this process must have the form

$$T=\int(dx)\Phi_{p'}{}^{\rm out}(x)^*e^{-i\Delta\cdot x}M(\Delta,\cdots)$$
$$=\int(dx)\exp\left[-i\int_y^\infty dy' I_{p'}(y')\right]$$
$$\times e^{i(p'-\Delta)\cdot x}M(\Delta,\cdots), \quad (4.1)$$

where $M(\Delta,\cdots)$ is essentially the transition amplitude for the process in the absence of the laser beam. It depends upon the total energy-momentum transfer Δ to the other particles involved in the process, as well as upon other energy-momentum variables which we need not consider explicitly. Of course, the wave function which appears here is not gauge invariant. There must be other charged particles either in the initial or final states. However, we shall assume that we can choose a gauge in which the corresponding exponential factors for the other particles are near unity. This will be the case if these particles are slowly moving, and we use the radiation gauge. Clearly, we could consider simultaneously the effect of other charged particles, but this would merely make our treatment cumbersome and not essentially alter our conclusions.

We proceed now as in the previous section. We introduce a Fourier series decomposition of the exponential factor and then perform the integration over x to obtain

$$T=\textstyle\sum_r e^{ir\alpha}C_r(\xi,\eta,\beta)(2\pi)^4\delta(p'-\Delta-[r-\zeta]k)M, \quad (4.2)$$

where, as can immediately be inferred from the discussion of the previous section,

$$\xi e^{i\alpha}=e(p'\cdot\mathfrak{a}/p'\cdot k), \quad (4.3)$$

$$\eta e^{i(2\alpha+\beta)}=(e^2/4)(-\mathfrak{a}^2)(1/p'\cdot k), \quad (4.4)$$

$$\zeta=(e^2/4)(-\mathfrak{a}\cdot\mathfrak{a}^*)(1/p'\cdot k)>0. \quad (4.5)$$

Thus we have obtained a decomposition of the transition amplitude into an infinite sum of incoherent amplitudes. Since we are considering processes involving particles of high momenta, in general these momenta cannot be measured with sufficient accuracy

[15] The nonrelativistic limit is understood to imply $\omega/m=\omega'/m=0$ but not $\mathfrak{a}/m=0$. Note also that it does not imply $\omega=\omega'$. If one sets $\eta=\zeta=0$, but retains the contribution of ξ, this nonrelativistic limit becomes essentially that of Z. Fried [Phys. Letters 3, 349 (1963)]. However, ξ^2 is of the same order of magnitude as η and ζ, so that this procedure is not self-consistent.

[16] It is easy to see that the nonrelativistic limit in the general polarization case is also given by (3.42).

[17] We should observe that the parameter

$$(e\mathfrak{a}/\omega)^2\sim e^2(\mathcal{E}\lambda^3)\lambda$$

diverges as λ increases without bound while $\mathcal{E}\lambda^3$, which is proportional to the total energy of the field, remains finite (compare footnote 14). Thus it can have no direct physical significance.

to distinguish among the various harmonics. The transition rate will therefore be given by

$$\Gamma=\int \frac{(d\mathbf{p}')}{(2\pi)^3}\frac{1}{2p_0'}\sum_r |C_r|^2\gamma(p'-[r-\zeta]k)\,, \quad (4.6)$$

where

$$\gamma(p'-[r-\zeta]k) = \int (df)(2\pi)^4\delta(p'-[r-\zeta]k-\Delta)|M|^2\,, \quad (4.7)$$

and $\int(df)$ denotes the integration over the momenta of the other particles in the final state. The ranges of the various momentum integrations are of course limited and must correspond to the experimental situation. Clearly, $\gamma(p'-[r-\zeta]k)$ is the differential transition probability for the emission of a scalar particle of energy-momentum $p'-[r-\zeta]k$ in the absence of the laser beam, where, however, the argument $p'-[r-\zeta]k$ does not necessarily lie on the mass hyperboloid.

Now, if the velocity of the emitted scalar particle is large, we may, as a first approximation, neglect the $(r-\zeta)k$ term of the argument of γ, for the functions C_r decrease rapidly with increasing index r. It follows from the integral representation (3.11) of the C_r functions that

$$\sum_r |C_r(\xi,\eta,\beta)|^2 = \int_{-\pi}^{\pi}\frac{d\theta_1 d\theta_2}{(2\pi)^2}\sum_r e^{ir(\theta_1-\theta_2)} \times\exp\{i\xi[\sin\theta_1-\sin\theta_2] +\tfrac{1}{2}i\eta[\sin(2\theta_1-\beta)-\sin(2\theta_2-\beta)]\} =1\,, \quad (4.8)$$

since the sum over r yields, within the integration range, $2\pi\delta(\theta_1-\theta_2)$. The C_r functions are of course the Fourier components of the wave function corresponding to the various modes of propagation. The relation (4.8) above exhibits a closure property of these components. Hence in this first approximation, the transition rate is the same as that in the absence of the laser beam,

$$\Gamma=\Gamma_0. \quad (4.9)$$

Thus the presence of the laser beam has no large effect.

In order to consider more closely the order of magnitude of the effect, we write

$$\gamma(p)=\int (dx)e^{ip\cdot x}\gamma(x)\,. \quad (4.10)$$

The Fourier transform $\gamma(x)$ is appreciably different from zero only in the region $|x|\lesssim 1/E$, where E is roughly the largest energy available to the emitted scalar particle. For the processes of interest to us in which a fast particle can be emitted, this energy will be of the order of the particle mass, $E\approx m$. One may easily derive, using the method outlined above for obtaining the closure property (4.8), the more general relation

$$\sum_r |C_r(\xi,\eta,\beta)|^2 e^{-ir\varphi} =C_0(2\xi\sin\tfrac{1}{2}\varphi,\, 2\eta\sin\varphi,\, \beta+\tfrac{1}{2}\pi)\,. \quad (4.11)$$

Thus, on introducing the Fourier transform (4.10) of $\gamma(p)$ into the rate formula (4.6) we secure

$$\Gamma=\int \frac{(d\mathbf{p}')}{(2\pi)^3}\frac{1}{2p_0'}\int (dx)\gamma(x)e^{i(p'+\zeta k)\cdot x} \times C_0(2\xi\sin\tfrac{1}{2}k\cdot x,\, 2\eta\sin k\cdot x,\, \beta+\tfrac{1}{2}\pi)\,. \quad (4.12)$$

Now, since $\gamma(x)$ is small for $|x|>1/m$, we have, within the important integration domain,

$$|k\cdot x|\lesssim \omega/m \ll 1\,.$$

Hence, the effective parameters in the problem are not ξ and η, but the much smaller quantities

$$\xi\omega/m=O(e\mathfrak{A}/m)\,,\quad \eta\omega/m=O[(e\mathfrak{A}/m)^2]\,,$$

and these are of the same order of magnitude as those which occur in the case of Compton scattering. We have now achieved a form in which an expansion in these small parameters may be performed. We shall merely remark that such an expansion indicates that the most pronounced effects would occur in regions of rapid variation of the energy spectrum of the emitted particle.

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APPENDIX A

The electron Green's function in the presence of an external plane-wave field occurs in any process involving virtual electrons within a laser beam. In particular, we shall make use of it in the following Appendix where we discuss the vacuum-vacuum transformation function for a plane-wave field. We present here a direct and simple calculation of this function, and obtain a form which proves convenient for our later discussion.

The Green's function is defined as

$$G(x,x';A)=i\frac{\langle 0\text{ out}|T(\psi(x)\bar\psi(x'))|0\text{ in}\rangle_A}{\langle 0\text{ out}|0\text{ in}\rangle_A}\,, \quad (A1)$$

where T indicates the usual time-ordered product. It satisfies the equation

$$[m-\gamma\cdot\Pi]G(x,x';A)=\delta(x-x'), \qquad (A2)$$

and the associated boundary conditions that it contain only positive frequency components as x_0 or x_0' tend to $+\infty$, and only negative frequency components as x_0 or x_0' tend to $-\infty$. We shall solve for the Green's function using methods which are completely analogous to those used in the construction of the wave functions in Sec. 2. We set

$$G(x,x';A)=[\gamma\cdot\Pi+m]\mathcal{G}(x,x';A), \qquad (A3)$$

so that $\mathcal{G}$ satisfies the second-order equation

$$[-\Pi^2+m^2-\tfrac{1}{2}e\sigma F(x)]\mathcal{G}(x,x';A)=\delta(x-x'), \qquad (A4)$$

and has the same boundary conditions as G. We look for a solution of the form

$$\mathcal{G}(x,x';A)=\int\frac{(dp)}{(2\pi)^4}\frac{e^{-ip\cdot(x-x')}}{m^2-p^2-i\epsilon}f_p(y,y'),$$

where as before $y=n\cdot x$, $y'=n\cdot x'$. Then the equation for f_p becomes

$$\int\frac{(dp)}{(2\pi)^4}\frac{e^{-ip\cdot(x-x')}}{m^2-p^2-i\epsilon}2n\cdot p\left[-i\frac{\partial}{\partial y}+J_p(y)\right]f_p(y,y')$$
$$=\int\frac{(dp)}{(2\pi)^4}e^{-ip\cdot(x-x')}[1-f_p(y,y')], \qquad (A5)$$

where

$$J_p(y)=\frac{1}{2n\cdot p}[2ep\cdot A(y)-e^2A^2(y)-\tfrac{1}{2}e\sigma F(y)]$$
$$=I_p(y)-\frac{1}{4n\cdot p}e\sigma F(y). \qquad (A6)$$

The left-hand side of Eq. (A5) vanishes if we take

$$f_p(y,y')=\exp\left[-i\int_{y'}^{y}d\bar{y}J_p(\bar{y})\right].$$

To verify that the right-hand side also vanishes, we separate the integration over the component of p_μ in the direction of n_μ from the remainder. We write

$$p_\mu=\tilde{p}_\mu+\kappa n_\mu,$$

where $\tilde{p}_\mu$ is required to lie on a three-dimensional surface. Then $f_p(y,y')$ is independent of κ, and the integration over κ gives

$$\int\frac{d\kappa}{2\pi}e^{-i\kappa(y-y')}[1-f_p(y,y')]=\delta(y-y')[1-f_p(y,y')]=0.$$

This demonstrates that we have obtained a solution of the inhomogeneous second-order Dirac equation, namely

$$\mathcal{G}(x,x';A)=\int\frac{(dp)}{(2\pi)^4}\frac{1}{m^2-p^2-i\epsilon}$$
$$\times\exp\left[-ip\cdot(x-x')-i\int_{y}^{y}d\bar{y}J_p(\bar{y})\right]. \qquad (A7)$$

One may readily verify, by performing the usual contour integral over p_0 that, for a field corresponding to a finite wave train, the solution we have derived satisfies the correct boundary conditions.

We now wish to write this result in a form in which the dependence of the integrand on p is simple. To do this, we make a formal translation of the integration variable,[18] by defining

$$p_\mu'=p_\mu+\frac{n_\mu}{y-y'}\int_{y'}^{y}d\bar{y}J_p(\bar{y})-\frac{1}{y-y'}\int_{y'}^{y}d\bar{y}eA_\mu(\bar{y}),$$

so that the exponent is

$$-ip\cdot(x-x')-i\int_{y'}^{y}d\bar{y}J_p(\bar{y})=-ip'\cdot(x-x')-ie\Lambda(x,x'),$$

where

$$\Lambda(x,x')=\frac{(x-x')}{y-y}\cdot\int_{y'}^{y}d\bar{y}A(\bar{y}). \qquad (A8)$$

One readily verifies that the Jacobian of this transformation is unity, and that since $n\cdot p'=n\cdot p$ the denominator may be written as

$$-p^2+m^2=-p'^2+m^2+\mathfrak{M}^2(y,y')-\frac{1}{y-y'}\int_{y'}^{y}d\bar{y}\tfrac{1}{2}e\sigma F(\bar{y}),$$

where

$$\mathfrak{M}^2(y,y')=-\frac{1}{y-y'}\int_{y'}^{y}d\bar{y}e^2A^2(\bar{y})$$
$$+\left(\frac{1}{y-y'}\int_{y'}^{y}d\bar{y}eA(\bar{y})\right)^2. \qquad (A9)$$

Recalling that $\sigma F(y)\sigma F(y')=0$, we see that the term involving σF may be removed from the denominator by expanding to first order in σF. Since A_μ is a space-like vector, it follows from the Schwartz inequality that

$$\mathfrak{M}^2(y,y')=(y'-y)^{-2}\left\{-\int_{y'}^{y}d\bar{y}e^2A^2(\bar{y})\int_{y'}^{y}d\bar{y}\right.$$
$$\left.+\left[\int_{y'}^{y}d\bar{y}eA(\bar{y})\right]^2\right\}\geqslant 0.$$

[18] The translation of an integration variable by a quantity involving a matrix may in general be accomplished by diagonalizing the matrix and translating the integration variable by its various eigenvalues. In the present case, the square of σF vanishes, and it therefore cannot be diagonalized. However, since the integral is a continuous function of the elements of this matrix, we may infinitesimally alter $F_{\mu\nu}$ in such a way that σF can be brought to diagonal form, and then let $F_{\mu\nu}$ approach its true value after performing the integration.

In particular, for a monochromatic plane wave, we have, for $|\omega(y-y')|\gg 1$,

$$\mathfrak{M}^2(y,y')\sim\Delta m^2=\tfrac{1}{2}e^2(-\mathfrak{a}\cdot\mathfrak{a}^*)>0\,, \qquad \text{(A10)}$$

which shows that over long times an electron propagates in the laser beam with the altered mass value $m^2+\Delta m^2$. Because of the positive definite character of $\mathfrak{M}^2(y,y')$, there is no difficulty in writing our result in the explicit form

$$\mathcal{G}(x,x';A)=e^{-ie\Lambda(x,x')}\left\{1-\frac{1}{y-y'}\int_{y'}^{y}d\bar{y}\tfrac{1}{2}e\sigma F(\bar{y})\frac{\partial}{\partial m^2}\right\}$$
$$\times\Delta_c[x-x';m^2+\mathfrak{M}^2(y,y')]\,, \qquad \text{(A11)}$$

where Δ_c is the familiar free-field Green's function,

$$\Delta_c(x;m^2)=\int\frac{(dp)}{(2\pi)^4}\frac{e^{-ip\cdot x}}{m^2-p^2-i\epsilon}\,. \qquad \text{(A12)}$$

We note that, since $n\cdot A=0$, the function $\mathfrak{M}^2(y,y')$ is invariant under a gauge transformation

$$A_\mu(y)\to A_\mu(y)+\partial_\mu\lambda(y)=A_\mu(y)+n_\mu(d/dy)\lambda(y)\,.$$

Thus the entire gauge dependence of $\mathcal{G}$ is contained in $\Lambda(x,x')$. This function may be expressed as the straight line integral

$$\Lambda(x,x')=\int_{x'}^{x}d\bar{x}^\mu A_\mu(\bar{x})$$
$$=\int_0^1 d\tau(x-x')\cdot A[y'+(y-y')\tau]\,, \qquad \text{(A13)}$$

which explicitly shows that under a gauge transformation,

$$\Lambda(x,x')\to\Lambda(x,x')+\lambda(y)-\lambda(y')\,.$$

It is convenient to write the first-order Green's function also in a form in which its entire gauge dependence is isolated in the exponential factor involving Λ. Accordingly, we define

$$\tilde{\Pi}_\mu=e^{ie\Lambda(x,x')}\Pi_\mu e^{-ie\Lambda(x,x')}$$
$$=i\partial_\mu-e\tilde{A}_\mu(x,x')\,. \qquad \text{(A14)}$$

One easily verifies that $\tilde{A}_\mu$ may be written in the gauge invariant from

$$\tilde{A}_\mu(x,x')=\int_0^1 d\tau\tau F_{\mu\nu}[y'+(y-y')\tau](x-x')^\nu. \qquad \text{(A15)}$$

Thus, on writing the integral involving σF in a similar form, we achieve the structure

$$G(x,x';A)=e^{-ie\Lambda(x,x')}[\gamma\cdot\tilde{\Pi}+m]$$
$$\times\left[1-\int_0^1 d\tau\tfrac{1}{2}e\sigma F[y'+(y-y')\tau]\frac{\partial}{\partial m^2}\right]$$
$$\times\Delta_c[x-x';m^2+\mathfrak{M}^2(y,y')]\,. \qquad \text{(A16)}$$

APPENDIX B

In the text we have made use of the fact that, in our approximation of neglecting radiative corrections, the vacuum state is unchanged by an external plane-wave field, or, equivalently, that the vacuum-vacuum transformation function is unity,

$$\langle 0\text{ out}|0\text{ in}\rangle_A=1\,. \qquad \text{(B1)}$$

This result is to be expected on physical grounds, for it is clear that a plane-wave field can only change the momentum by some multiple of the null vector n_μ, so that from the vacuum it cannot create electron-positron pair states, whose mass is greater than zero. It is the purpose of this Appendix to establish the result (B1) rigorously for the general case of an arbitrarily polarized plane-wave field using the formalism we have developed. In the course of our discussion, we shall encounter the problem of finding a correct definition of the current operator in quantum field theory. We hope, therefore, that our considerations may shed some light on this problem and be of some general interest.

The vacuum current induced by an external field is given by the response of the vacuum-vacuum transformation function to variations of this field,

$$\langle j^\mu(x)\rangle_A=\frac{\langle 0\text{ out}|j^\mu(x)|0\text{ in}\rangle_A}{\langle 0\text{ out}|0\text{ in}\rangle_A}$$
$$=i\frac{\delta}{\delta A_\mu(x)}\ln\langle 0\text{ out}|0\text{ in}\rangle_A\,. \qquad \text{(B2)}$$

To establish the result (B1) it is therefore necessary and sufficient to show that in a plane-wave field the vacuum current $\langle j^\mu(x)\rangle_A$ vanishes identically. It is usually assumed that the current operator which appears in (B2) is the bilinear combination of Fermi field operators

$$j_{(0)}{}^\mu(x)=\tfrac{1}{2}e\psi(x)\gamma^0\gamma^\mu q\psi(x)\,, \qquad \text{(B3)}$$

but this is a very singular object, which requires careful definition.

Here we have changed from our previous non-Hermitian fields to Hermitian fields. The use of Hermitian fields is often very convenient for the consideration of the various invariance properties of field theory, but we employ them here simply for their convenience in eliminating the necessity of ordering the field operators, and the resulting facility of computation. The new Hermitian fields have an additional two-dimensional multiplicity characterizing the charge they bear, and are related to the previous non-Hermitian fields by

$$\psi=\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}(\psi+\psi^\dagger)\\ i(\psi-\psi^\dagger)\end{bmatrix}. \qquad \text{(B4)}$$

The Green's function for these fields is simply obtained from that of the previous Appendix by replacing the electrical charge e by the charge matrix

$$eq=e\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{B5}$$

We must now use a Majorana representation of the Dirax matrices, in which the matrices $(\gamma^0\gamma^\mu)$ are real and symmetric.

The bilinear combination (B3) of Fermi field operators must be defined as some limit in which the two field coordinates are coalesced. Since gauge invariance should be maintained at all stages, we shall define this operator as[19]

$$j^\mu{}_{(0)}(x)=\operatorname*{symm\,lim}_{\epsilon\to 0} \tfrac{1}{2}e\bar\psi(x+\tfrac{1}{2}\epsilon)\gamma^0\gamma^\mu q \times\exp[-ieq\Lambda(x+\tfrac{1}{2}\epsilon,\, x-\tfrac{1}{2}\epsilon)]\psi(x-\tfrac{1}{2}\epsilon), \tag{B6}$$

where $\Lambda(x',x'')$ is the line integral of the vector potential defined in (A13). As we have indicated, the limit of vanishing ϵ is to be taken in a symmetrical fashion. Since we wish to preserve the canonical, Hamiltonian nature of the theory, we shall average only over spatial values of ϵ. Thus

$$\langle j^\mu{}_{(0)}(x)\rangle_A=\operatorname*{symm\,lim}_{x',x''\to x} (-\tfrac{1}{2}e)\ \mathrm{Tr}q\gamma^\mu e^{-ieq\Lambda(x',x'')} \times(1/i)G(x'',x';A), \tag{B7}$$

where the trace is over both the spinor and charge indices. On inserting the expression (A16) for the Green's function into this relation we see that the two line integrals cancel, and, after performing the trace, we secure

$$\langle j^\mu{}_{(0)}(x)\rangle_A=\operatorname*{symm\,lim}_{x',x''\to x} 4e^2\left\{\tilde A^\mu(x'',x') +\partial_\nu''\int_0^1 d\tau F^{\mu\nu}(y'+\tau[y''-y'])\frac{\partial}{\partial m^2}\right\} (1/i)\Delta_c[x''-x';m^2+\mathfrak{M}(y'',y')]. \tag{B8}$$

The derivative ∂_ν'' is effective only when it acts on the coordinate difference $x''-x'$ occurring in Δ_c, for otherwise it gives terms proportional to n_ν, and $F^{\mu\nu}n_\nu=0$. For a space-like separation of points, this derivative of the Δ_c function is odd under the interchange of y' and y'', while the term

$$\int_0^1 d\tau F^{\mu\nu}(y'+\tau[y''-y'])$$

is even under this interchange. Thus the second term in the square brackets above does not contribute in the symmetrical limit, and we need consider only the first term. On expanding this term about the point x we have

$$\langle j^\mu{}_{(0)}(x)\rangle_A=\operatorname*{symm\,lim}_{\epsilon\to 0} 4e^2[-\tfrac{1}{2}\epsilon_\nu F^{\mu\nu}(y) +(1/12)\epsilon^\lambda\epsilon_\nu\partial_\lambda F^{\mu\nu}(y)+0(\epsilon^3)] \times(1/i)\Delta_c[-\epsilon;m^2+\mathfrak{M}^2(y'',y')].$$

Now for a space-like argument

$$(1/i)\Delta_c(x;m^2)=[m/4\pi^2(-x^2)^{1/2}]K_1[m(-x^2)^{1/2}] =1/4\pi^2\{1/(-x^2)+(m^2/4) \times[\ln m^2(-x^2)+O(1)]\}. \tag{B9}$$

Thus

$$\langle j^\mu{}_{(0)}(x)\rangle_A=-(e^2/4\pi)\partial_\lambda F^{\mu\nu}(y)\ \operatorname*{symm\,lim}_{\epsilon\to 0}\ \epsilon^\lambda\epsilon_\nu/3\pi\epsilon^2,$$

and, on averaging over all spatial directions, we find the noncovariant and nonvanishing result[20]

$$\langle j^\mu{}_{(0)}(x)\rangle_A=-(e^2/4\pi)(1/9\pi)\partial_k F^{\mu k}(y). \tag{B10}$$

Here and in what follows, latin indices range over only the three spatial coordinates.

This difficulty is removed when it is realized that we must consider the variation of the complete Hamiltonian density in obtaining the vacuum current, for it is this function that specifies the time-development of the vacuum state. We define[21] this function as

$$\mathfrak{H}(x)=\operatorname*{symm\,lim}_{\epsilon\to 0} \tfrac{1}{2}\bar\psi(x+\tfrac{1}{2}\epsilon)\gamma^0 e^{-ieq\Lambda(x+\frac{1}{2}\epsilon,x)} \times[-i\gamma^k\partial_k+eq\gamma^\mu A_\mu(x)+m] \times e^{-ieq\Lambda(x,x-\frac{1}{2}\epsilon)}\psi(x-\tfrac{1}{2}\epsilon). \tag{B11}$$

Accordingly, the current operator is given by

$$j^\mu(x)=\frac{\delta}{\delta A_\mu(x)}\int(d\bar x)\mathfrak{H}(\bar x) =j^\mu{}_{(0)}(x)+j^\mu{}_{(1)}(x), \tag{B12}$$

where $j^\mu{}_{(1)}$ arises from the variation of the line integrals. Now

$$\frac{\delta}{\delta A_\mu(x)}\Lambda(\bar x+\tfrac{1}{2}\epsilon;\bar x)=\int_0 d\tau\tfrac{1}{2}\epsilon^\mu\delta(\bar x+\tfrac{1}{2}\epsilon\tau-x),$$

and we can therefore use the δ function to perform the integration over $\bar x$. It is not hard to verify, using the symmetry properties of the Majorana-Dirac and charge matrices, that the variation of the second line integral

[19] This definition was proposed by J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

[20] If we transform to a Euclidean space-time world [see J. Schwinger, Phys. Rev. 115, 721 (1959)], perform the averaging procedure, and then transform back to Minkowskian space, we find a vanishing result, since $\partial_\nu F^{\mu\nu}=0$. This procedure, however, violates the canonical basis of the theory.

[21] The necessity of the introduction of exponential line integral terms in the Hamiltonian has been noted before in the study of a simple model, L. S. Brown, Nuovo Cimento 29, 617 (1963).

yields a contribution which differs from that of the first line integral only in the sign of ϵ. In the symmetric limit these terms contribute equally, and we obtain

$$j^\mu{}_{(1)}(x) = -\tfrac{1}{2}ie \operatorname{symm}\lim_{\epsilon\to 0}\int_0^1 d\tau\epsilon^\mu\bar\psi(\bar x')\gamma^0 q \times e^{-ieq\Lambda(\bar x',\bar x)}[-i\gamma^k(\bar\partial_k+\bar\partial_k'')+eq\gamma^\nu A_\nu(\bar x)+m] \times e^{-ieq\Lambda(\bar x,\bar x'')}\psi(\bar x''), \quad \text{(B13)}$$

where

$$\bar x^\mu = x^\mu - \tfrac{1}{2}\epsilon^\mu\tau;\quad \bar x'^\mu = \bar x^\mu + \tfrac{1}{2}\epsilon^\mu;\quad \bar x''^\mu = \bar x^\mu - \tfrac{1}{2}\epsilon^\mu.$$

It is convenient to write the vacuum expectation of this operator in a form in which only the gauge-invariant part of the Green's function appears. After some manipulation we obtain

$$\langle j^\mu{}_{(1)}(x)\rangle_A = \tfrac{1}{2}ie \operatorname{symm}\lim_{\epsilon\to 0}\int_0^1 d\tau\epsilon^\mu \operatorname{Tr}q\{-i\gamma^k\bar\partial_k'' + eq\gamma^k[\tilde A_k(\bar x,\bar x'')-\tilde A_k(\bar x'',\bar x) + \tilde A_k(\bar x'',\bar x')]+eq\gamma^0 A_0(\bar x)+m\} \times e^{-ieq\Lambda(\bar x',\bar x'')}(1/i)G(\bar x'',\bar x';A). \quad \text{(B14)}$$

We now examine the various terms in the curly brackets. The trace over the term involving m vanishes identically when the explicit expression (A16) is substituted for the Green's function. The term in A_0 does not contribute for purely spatial ϵ, since the singular terms in the spinor trace of $\gamma^\mu G$ are proportional to ϵ^μ. Next, we consider the expression in square brackets. It may be expanded about the point x, and the leading term is

$$-(1/24)\epsilon^l\partial_l F_{km}(y)\epsilon^m.$$

However, the leading term in the spinor trace of $\gamma^k G$ is proportional to $\epsilon^k/(-\epsilon^2)^2$, and therefore gives a vanishing contribution because of the antisymmetry of F_{km}. The remaining term involves precisely the trace which occurs in the calculation of the usual current contribution (B7). Thus, recalling the evaluation of this trace (B8), we obtain

$$\langle j^\mu{}_{(1)}(x)\rangle_A = -4e^2 \operatorname{symm}\lim_{\epsilon\to 0}\int_0^1 d\tau\epsilon^\mu\bar\partial_k''\left\{\tilde A^k(\bar x'',\bar x') + \bar\partial_\nu''\int_0^1 d\bar\tau F^{k\nu}(\bar y'+\bar\tau[\bar y''-\bar y'])\frac{\partial}{\partial m^2}\right\} \times (1/i)\Delta_c[\bar x''-\bar x';m^2+\mathfrak{M}^2(\bar y'',\bar y')]. \quad \text{(B15)}$$

As we remarked before, the derivative $\bar\partial_\nu''$ is effective only when it acts on the coordinate difference in the Δ_c function, and thus gives terms proportional to ϵ_ν. On the other hand, the derivative $\bar\partial_k$ is not effective when it acts on this coordinate difference, for recalling the definition (A15) of $\tilde A^k(\bar x'',\bar x')$, we see that the resulting contributions from both terms in the curly brackets involve the expression $\epsilon_k F^{k\nu}\epsilon_\nu$, which vanishes for purely spatial ϵ. Moreover, it is easy to verify that when $\bar\partial_k''$ acts on the argument of the function $\mathfrak{M}^2$ it yields only terms which vanish in the limit $\epsilon\to 0$. Thus it is effective only when acting on the terms inside the curly brackets. Expanding these terms about the point x we obtain

$$\bar\partial_k''\tilde A^k(\bar x'',\bar x') = -\tfrac{1}{3}\partial_k F^{k\nu}(y)\epsilon_\nu + O(\epsilon^2),$$

$$\bar\partial_k''\int_0^1 d\bar\tau F^{k\nu}(\bar y'+\bar\tau[\bar y''-\bar y']) = \tfrac{1}{2}\partial_k F^{k\nu}(y)+O(\epsilon).$$

Hence, inserting the form (B9) for the Δ_c function of small argument, we find

$$\langle j^\mu{}_{(1)}(x)\rangle_A = -4e^2 \operatorname{symm}\lim_{\epsilon\to 0}\epsilon^\mu \times\left\{-\tfrac{1}{3}\partial_k F^{k\nu}(y)\epsilon_\nu\frac{1}{4\pi^2(-\epsilon^2)} + \tfrac{1}{2}\partial_k F^{k\nu}(y)\frac{\epsilon_\nu}{8\pi^2(-\epsilon^2)}\right\}$$

$$= -\left(\frac{e^2}{4\pi}\right)\partial_k F^{k\nu}(y) \operatorname{symm}\lim_{\epsilon\to 0}\frac{\epsilon^\mu\epsilon_\nu}{3\pi\epsilon^2}.$$

Thus, on performing the spatial average, and recalling that $\partial_k F^{k0}$ vanishes, we have

$$\langle j^\mu{}_{(1)}(x)\rangle_A = -\left(\frac{e^2}{4\pi}\right)\partial_k F^{k\mu}(y)\frac{1}{9\pi}, \quad \text{(B16)}$$

which precisely cancels the usual vacuum current contribution (B10). Thus, finally, we have proved that the total vacuum current is zero:

$$\langle j^\mu(x)\rangle_A = 0. \quad \text{(B17)}$$

This is in fact a particular case of a more general result, valid for arbitrary external fields. In the general case, a similar procedure to that adopted here is necessary to obtain a covariant, though in general nonvanishing (and indeed logarithmically divergent), vacuum current. The details of this discussion will be published elsewhere.

APPENDIX C

We have noted in connection with Eq. (3.31) that the parameter ν^2 is of a purely classical nature. The nonrelativistic cross sections, Eq. (3.42), are obtained by neglecting terms of order $\hbar\omega/m$, and, since $\hbar$ does not appear elsewhere, the nonrelativistic limit must actually coincide with the classical limit. Vachaspati[3a] has obtained similar, though not identical, results to our Eqs. (3.43) and (3.44) on the basis of classical electrodynamics. We should like to clarify here the classical nature of the principal results of Sec. 3, and the reasons for the discrepancy between Vachaspati's results and ours.

One can solve directly the classical equations of motion for a particle in an electromagnetic plane-wave field. However, instead of doing this, we shall use a covariant generalization of the Hamilton-Jacobi method, which is mathematically simpler, and corresponds more closely with the quantum-mechanical treatment of Sec. 3.

In the classical limit, we may neglect spin terms, and therefore consider only scalar particles. The wave function of Eq. (2.6) is of the form

$$\Phi_p{}^{\rm in}(x)=\exp[-iS_p(x)/\hbar],$$

where

$$S_p(x)=p\cdot x+\int_{-\infty}^{y}dy'I_p(y')$$

is independent of $\hbar$. This demonstrates that the WKB approximation is exact in this situation.

The function $S_p(x)$ satisfies the equation

$$[\partial_\mu S_p(x)-eA_\mu(x)]^2=p^2,$$

which is easily recognizable as the covariant proper-time form of the Hamilton-Jacobi equation. Hence $S_p(x)$ is Hamilton's characteristic function. The four quantities p_μ here play the role of integration constants determined by the initial conditions. According to standard Hamilton-Jacobi theory, the classical position $x_\mu(\tau)$ of the particle is obtained as a function of its proper time by differentiating with respect to these integration constants. This yields

$$(\partial/\partial p_\mu)S_p(x(\tau))=x^\mu{}_{(0)}+(p^\mu/m)\tau,$$

where $x^\mu{}_{(0)}$ is a further constant four-vector. Explicitly, we find

$$x^\mu(\tau)-\frac{n^\mu}{n\cdot p}\int_{-\infty}^{y(\tau)}dy'I_p(y')+\frac{e}{n\cdot p}\int_{-\infty}^{y(\tau)}dy'A^\mu(y')=x^\mu{}_{(0)}+\frac{p^\mu}{m}\tau,$$

whence, in particular,

$$y(\tau)=y_{(0)}+\frac{n\cdot p}{m}\tau.$$

The classical canonical momentum is given by the derivative with respect to $x^\mu(\tau)$,

$$p_\mu(\tau)=m\dot{x}_\mu(\tau)+eA_\mu[x(\tau)]=\partial_\mu S_p(x(\tau)),$$

or, in our case,

$$p^\mu(\tau)=p^\mu+n^\mu I_p[y(\tau)].$$

It follows that the time average of the momentum over many oscillations of the electromagnetic wave is

$$\langle p^\mu(\tau)\rangle_{\rm av}=\langle m\dot{x}^\mu(\tau)\rangle_{\rm av}=p^\mu+(n^\mu/2n\cdot p)\langle -e^2A^2[y(\tau)]\rangle_{\rm av}=p^\mu+(m^2\nu^2/2n\cdot p)n^\mu,$$

where ν^2 is defined by a generalized form of Eq. (3.27). This is precisely the "effective momentum inside the beam" $\bar{p}^\mu$ of Eq. (3.16). Classically, therefore, the increased effective mass of the electron may be interpreted as arising from its oscillatory motion induced by the field.

The simplest way of obtaining the classical cross sections is to use the Fourier transform of the classical current

$$j^\mu(x')=e\int d\tau\dot{x}^\mu(\tau)\delta[x'-x(\tau)].$$

The procedure is closely analogous to the quantum-mechanical reduction procedure employed in the text. The outgoing wave vector potential $A'_\mu(x)$ produced by the current $j_\mu(x)$ is

$$A_\mu'(x)=\int(dx')D_{\rm ret}(x-x')j_\mu(x').$$

At large times, we may expand $A'(x)$ in terms of a complete set of solutions of the free wave equation (say plane waves), and each coefficient in this expansion is given by the usual three-dimensional surface integral over $A'(x)$. By adjoining a surface in the remote past, where A' vanishes, we can convert this into a four-dimensional volume integral. The amplitude for each plane wave component is just the appropriate Fourier component of the current,

$$\int(dx)e^{ik'\cdot x}\epsilon'\cdot j(x)=e\int d\tau e^{ik'\cdot x(\tau)}\epsilon'\cdot\dot{x}(\tau).$$

For a monochromatic field, the explicit expression for $x^\mu(\tau)$, which is easily obtained from the formulas above, contains a term linear in y (or τ), which gives rise to the frequency shift, and also terms in $\sin\omega y$ and $\sin2\omega y$. On inserting this expression into the Fourier transform of the current, we obtain an exponential structure similar to that of Eq. (3.5), which may again be expanded in terms of the C_r functions, using Eq. (3.9). The integration over τ (or y) yields a δ function expressing the relation between ω and ω', namely the nonrelativistic form of Eq. (3.34); and the coefficient of the δ function is (apart from trivial factors) the function A_r defined in Eq. (3.24). This procedure finally yields precisely the nonrelativistic cross sections given in Eq. (3.42).

Vachaspati[3a] has evaluated the classical cross sections to order ν^2, corresponding to our Eqs. (3.43) and (3.44). Our results differ from his because they are expressed in a different Lorentz frame. He considers a monochromatic beam from the outset, and uses a frame in which the electron is *on the average* at rest, that is the rest frame of $\bar{p}$; we regard a monochromatic beam as the limit of a wave train of finite duration, and use the laboratory frame in which the electron is at rest before the arrival of the beam, that is the rest frame of p. The wavelength shift arises, in fact, from the Lorentz transformation from the rest frame of $\bar{p}$ to the laboratory frame. It is therefore of crucial importance to distinguish clearly between these frames.

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High-Magnetic-Field Specific Heat of a Low-Dislocation-Density Alloy Superconductor*

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The specific heat C of a *well-annealed* alloy V-5 at.% Ta, measured at $1.4 \leq T \leq 5°$K in steady magnetic fields, displays sharp, bulk, superconducting transitions at upper critical fields H_{c2} a factor ≈ 10 larger than the calorimetrically derived thermodynamic critical fields H_c. The transitions are similar to those observed earlier by Morin *et al.*[1] in V_3Ga, but in the present case it is unlikely that the bulk nature of the high-field transitions can be attributed to a nearly complete occupation of the specimen volume by dislocation-centered high-field superconducting filaments of diameter comparable to the penetration depth, since electron transmission microscopy studies on an identically prepared specimen indicate that in at least 95% of the specimen volume the mean separation between dislocations is greater than 1.4×10^{-4} cm. However, the present data are explicable on the basis of the Ginzburg-Landau-Abrikosov-Gor'kov theory with a parameter $\kappa \approx H_{c2}/\sqrt{2}H_c \approx 7$. The transition specific heat jumps $\Delta C(T_s)/\gamma T_s = 1.44$, 1.15, 1.10, 0.94 occur at $T_s = 4.30$, 4.09, 3.85, 3.37°K in fields $H = 0$, 1, 2, 4 kG, respectively, where $\gamma \equiv$ normal state electronic specific heat coefficient $= 9.20$ mJ/mole(K°)^2. The $\Delta C(T_s)$ values are in fair agreement with those calculated via Ehrenfest's equation for second-order phase transitions using Abrikosov's theoretical value of $(\partial I/\partial H)_T$ at T_s for $\kappa = 7$, where $I \equiv$ magnetization. For $(T_s/T) \geq 1.8$, $C_{es}/\gamma T_s = a \exp(-bT_s/T)$ with $a = 8.95$, 6.24, 5.01, 4.7; $b = 1.48$, 1.28, 1.17, 1.1; for $H = 0$, 1, 2, 4 kG, respectively, where C_{es} is the electronic contribution to the specific heat. The exponential temperature dependence of C_{es} down to 1.4°K suggests an essentially everywhere finite, field-dependent, high-field energy gap in accord with Abrikosov's vortex model.

RECENT calorimetric measurements of Morin *et al.*[1] revealed bulk, reversible, second-order superconducting transitions in V_3Ga at magnetic fields much larger than the thermodynamic critical fields. The surprising bulk nature of the high-field transitions was attributed[1,2] to a nearly complete occupation of the specimen volume by dislocation-centered high-field superconducting *filaments* of diameter comparable to a penetration depth $\lambda \approx 5\times10^{-6}$ cm,[3] requiring a high dislocation density $\approx 4\times10^{10}$ cm^{-2}.[2] On the other hand, it was suggested[4,5] that the transitions in V_3Ga might be the thermodynamic manifestation of a "*second kind*" *of superconductivity* explicable on the basis of the spatially uniform negative-surface-energy theories of Ginzburg, Landau, Abrikosov, and Gor'kov (GLAG)[6–10] or of Goodman.[11,12] The present measurements offer strong support for the negative-surface-energy interpretation since high-field bulk-superconducting calorimetric transitions are observed for a solid-solution bcc superconducting alloy, V-5 at.% Ta, which, after arc melting, was annealed for 2 h at 1600°C (0.85 of the melting

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